ON THE PROBLEM OF BRINGING A LINEAR SYSTEM TO ITS EQUILIBRIUM POSITION

(K ZADACHE OB USPOKOENII LINEINOI SISTEMY)

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This paper describes a procedure based on the steepest descent method for solving the problem of the optimum returning to the origin of a control system.

4. We shall consider the control system described by the linear vector differential equation

$$\frac{dx}{dt} = Ax + Bu \tag{1.1}$$

where x represents an n-dimensional vector of the phase coordinates of the controlled object, and u is a scalar function describing the control signal.

The problem of the optimum control $u^\circ(t)$ which in a given time T brings the system (1.1) from the state x_0 to the state $x(T)$ with the requirement that the quantity

$$J(u) = \max\left\{\max_\tau |u(\tau)|,\ \theta\int_0^T |u(\tau)|\,d\tau\right\} = \min \qquad (\theta = \mathrm{const}) \tag{1.2}$$

is a minimum, can be considered as follows [1].

Find the numbers $l_i\ (i = 1, \ldots, n)$ and the system Δ of intervals $[\tau_k, \tau_{k+1}]$ on $[0,T]$ for which

$$\min_l \max_\Delta \int_\Delta \left|\sum_{i=1}^n l_i h_i(\tau)\right| d\tau = \gamma \tag{1.3}$$

is satisfied with the condition

$$\sum_{i=1}^n l_i c_i = 1, \qquad \mu(\Delta) = \min\left[\frac{1}{\theta},\ T\right]$$

where

$$h_i(\tau) = \sum_{j=1}^n f_{ij}(-\tau)\, b_j \quad (i = 1, \ldots, n) \qquad (c = -x_0)$$

The $f_{ij}(t)$ are the elements of the fundamentsl matrix $F(t)$ of the homogeneous system (1.1); $\mu(\Delta)$ is the overall length of the system of intervals $[\tau_k, \tau_{k+1}]$.

Once (1.3) has been solved, the optimum control $u^\circ(\tau)$ is determined by Equations

$$u^\circ(\tau) = \frac{1}{\gamma}\,\mathrm{sign}\sum_{i=1}^{n} l_i^\circ h_i(\tau) \quad \text{for } \tau \in \Delta^\circ \qquad u^\circ(\tau) = 0 \quad \text{for } \tau \,\overline{\in}\, \Delta^\circ \tag{1.4}$$

where l_i° $(i = 1, \ldots, n)$ and Δ° are solutions of (1.3)

We shall assume that the system (1.1) is fully controllable [2]. The quantity

$$\rho(l) = \max\nolimits_\Delta \int_\Delta \left|\sum_{i=1}^{n} l_i h_i(\tau)\right| d\tau, \qquad \mu(\Delta) = \min\left[\frac{1}{\theta},\ T\right] \tag{1.5}$$

is positive for all l_i $(i = 1, \ldots, n)$, which satisfy the condition $l_1^2 + l_2^2 + \ldots + l_n^2 > 0$. In the domain $\{l_i\}$ this quantity possesses the properties of a norm. Therefore, in order to find a minimum of (1.3), we can search for l_i° and Δ° by using the steepest descent method with respect to l_i.

2. In order to apply the steepest descent method to (1.3), it is necessary to compute the derivatives $\partial\rho/\partial l_i$ of $\rho(l)$. We shall calculate them, taking into consideration that the fuctions $h_i(\tau)$ which enter Equation (1.3) are very smooth. Let $c_n \neq 0$ for definiteness. Then, $\rho(l)$ can be formulated as

$$\rho(l) = \max\nolimits_\Delta \int_\Delta \left|\sum_{i=1}^{m} l_i g_i(\tau) + g_n(\tau)\right| d\tau \qquad (m = n-1) \tag{2.1}$$

where $g_i(\tau)$, $g_n(\tau)$ are known expressions of $h_i(\tau)$. We shall consider the case for which $\mu(\Delta) = 1/\theta < T$, since the derivatives $\partial\rho/\partial l_i$ have the form

$$\frac{\partial\rho}{\partial l_i} = \int_0^T g_i(\tau)\,\mathrm{sign}\Big(\sum_{j=1}^{m} l_j g_j(\tau) + g_n(\tau)\Big) d\tau \qquad (i = 1, \ldots, m) \tag{2.2}$$

when $\Delta = [0,T]$ and the descent of $\rho(l)$ with respect to l_i does not present any singularities. In (2.1), $\max_\Delta$ is obtained for the system of intervals $[\tau_k,\ \tau_{k+1}]$ $(k = 1, \ldots, s-1)$, located in the domain of the largest values of the function

$$w(\tau, l) = |g(\tau, l)|$$
$$g(\tau, l) = \sum_{i=1}^{m} l_i g_i(\tau) + g_n(\tau) \tag{2.3}$$

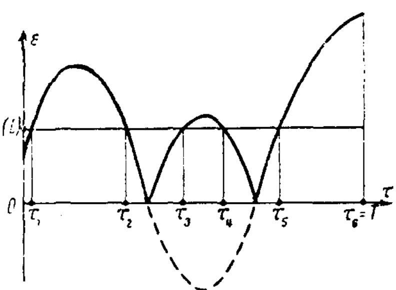

Fig. 1

On the ends of the intervals $\tau = \tau_k$ which do not coincide with $\tau = 0$ or $\tau = T$, the function (2.3) takes the equal values

$$w(\tau, l) = \varepsilon(l)$$

Let the numbers l_i be chosen in a certain manner, and a system of intervals $\Delta(l)$ be found for these values, such that it guarantees $\max_\Delta$ in (2.1). To define the problem, let us assume that $\tau = 0$ is not one of the points τ_k, but that $\tau = T$ is the point τ_s. Then we have the case shown on **Fig. 1**.

First, let us assume that a change Δl_i of l_i changes the values τ_k but does not yield new roots τ_k of Equation

$$w(\tau, l) = \varepsilon(l) \tag{2.4}$$

In that case, the changes $\Delta\rho$ and $\Delta\epsilon$ are described exactly, up to their higher order terms, by Equations

$$\Delta\rho = \Delta l_i \int\limits_{\Delta(l)} g_i(\tau, l) \operatorname{sign} g(\tau, l)\, d\tau \tag{2.5}$$

$$\Delta l_i \sum_{k=1}^{s^*} \frac{g_i(\tau_k) \operatorname{sign} g(\tau_k, l)}{|[g_\tau'(\tau, l)]_{\tau=\tau_k}|} - \Delta\varepsilon \sum_{k=1}^{s^*} \frac{1}{|[g_\tau'(\tau, l)]_{\tau=\tau_k}|} = 0 \tag{2.6}$$

(see Fig.2 on which $\tan \alpha = g_\tau'(\tau, l)$ and $\Delta l_i^\circ = \Delta l_i g_i(\tau) \operatorname{sign} g(\tau, l)$).

The symbol s^* in (2.6) means that the summation is made along all τ_k which do not coincide with the ends of the interval $[0,T]$. Equation (2.6) proceeds from the condition $\mu(\Delta) = 1/\theta$ = const . On the basis of (2.5) and (2.6) we get the following expressions for the partial derivatives:

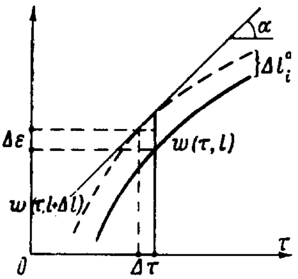

Fig. 2

$$\frac{\partial\rho}{\partial l_i} = \int\limits_{\Delta(l)} g_i(\tau, l) \operatorname{sign} g(\tau, l)\, d\tau \tag{2.7}$$

$$\frac{\partial\varepsilon}{\partial l_i} = \sum_{k=1}^{s^*} \frac{g_i(\tau_k) \operatorname{sign} g(\tau_k, l)}{|[g_\tau'(\tau, l)]_{\tau=\tau_k}|} \Big/ \sum_{k=1}^{s^*} \frac{1}{|[g_\tau'(\tau, l)]_{\tau=\tau_k}|}$$

Let us now consider the case in which Equation (2.4) gets new roots τ_k for arbitrarily small Δl_i. That case can occur only when the largest values of the function $w(\tau,l)$ are on the line $w = \varepsilon(l)$. First, let this occur for $\tau_1 = 0$, or $\tau_s = T$, whereupon $g_\tau'(\tau, l) \neq 0$, $\tau = \tau_1$ or $\tau = \tau_s$.

In such a case, if the condition

$$\Delta\varepsilon < \Delta l_i g_i(\tau_j) \operatorname{sign} g(\tau_j, l) \qquad (j = 1 \text{ or } j = s) \tag{2.8}$$

is fulfilled, then additional terms of the form

$$\frac{\Delta l_i g_i(\tau_j) \operatorname{sign} g(\tau_j, l)}{|[g_\tau'(\tau, l)]_{\tau=\tau_j}|} - \frac{\Delta\varepsilon}{|[g_\tau'(\tau, l)]_{\tau=\tau_j}|} \qquad (j = 1 \text{ or } j = s) \tag{2.9}$$

appear in Equation (2.6).

We shall note that if terms of the form (2.9) are considered in (2.6), it is indispensable to take into consideration the difference between the right and left values of the derivatives

$$\partial\varepsilon^+ / \partial l_i, \ \partial\varepsilon^- / \partial l_i$$

Let us now suppose that the largest value of the function $w(\tau,l)$ is found on the line $w = \varepsilon(l)$ for $\tau = \tau_j$ where τ_j is a point inside the interval $[0,T]$. We shall assume furthermore that $[g_\tau''(\tau_j, l)] \neq 0$, since the contrary would be an exceptional and not likely case. Then, terms of the form

$$\left(\frac{8(\Delta l_i g_i(\tau_j, l) \operatorname{sign} g(\tau_j, l) - \Delta\varepsilon)}{|g_\tau''(\tau_j, l)|}\right)^{1/2} \tag{2.10}$$

appear in Equation (2.6) with the condition that the positiveness of the radicand follows from (2.6), (where $\Delta l_i^\circ = \Delta l_i g_i(\tau)\,\mathrm{sign}\, g(\tau, l)$ (Fig.3)). Here again the difference between the right and the left derivatives must be taken into consideration. Terms of the form (2.10) are also found in (2.6) when $\tau_j = 0$ or $\tau_j = T$ and $g_\tau'(\tau, l) = 0$ for $\tau = \tau_j$. However, in that case, the factor 8 under the radical in the left-hand side of (2.10) is replaced by a factor 2. The values of the derivatives $\partial\rho/\partial l_i$ and the consideration of the remarks we made, determine the method to be used for the solution of problem (1.3), and also the problem of the optimum control by the system (1.1). Thus, as long as the largest values of the function $w(\tau, l)$ are sufficiently distant from the line $w = \varepsilon(l)$, the steepest descent of the quantity $\rho(l)$ determined by (2.1) must be obtained along the directions

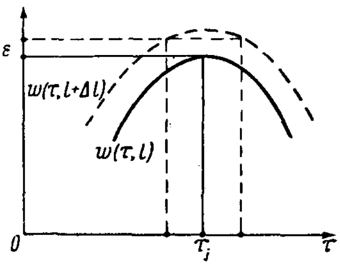

Fig. 3

$$\Delta l_i = -\nu \frac{\partial \rho}{\partial l_i}, \qquad \Delta\varepsilon = -\nu \sum_{i=1}^{n} \frac{\partial \varepsilon}{\partial l_i}\frac{\partial \rho}{\partial l_i}$$

where the derivatives

$$\partial\rho \,/\, \partial l_i, \qquad \partial\varepsilon \,/\, \partial l_i$$

are computed according to Formulas (2.7).

When values l_i, such that the largest values of the function $w(\tau, l)$ are in the neighborhood of the line $w = \varepsilon(l)$, are considered, one must be aware that new roots might appear (and similarly old roots disappear). Then, in the descent procedure, it is indispensable to bring in the corrections determined by these circumstances, and take into consideration terms of the form (2.9) and (2.10). Thus the steepest descent is determined by taking into account that the values $\partial\varepsilon^+/\partial l_i$ and $\partial\varepsilon^-/\partial l_i$ can be different.

In the cases in which the largest value of $w(\tau, l)$ is far from the line $w = \varepsilon(l)$, but $h_i'(\tau_k) = 0$, one must also consider terms of the form (2.10); however such cases are exceptional and we shall not discuss them. We should point out that the exposed method for determining the system of intervals Δ at each step of the calculation, has in the case of a numerical solution on a digital computer the disadvantage that it leads to a cumulation of errors. Therefore, when this method is used in a practical case, it is necessary, after a certain amount of steps, to check the conditions of conservation of the given measure of the system of intervals Δ.

This drawback can be avoided by the following method of approximate calculation for each fixed set of numbers l_i ($i = 1, \ldots, n$) of the system Δ of intervals $[\tau_k, \tau_{k+1}]$ of the specified measure for which a maximum of (2.1) is obtained, and which are necessary for the calculation of $\partial\rho/\partial l_i$ in agreement with (2.7) and also for the calculation of the quantity $\rho(l)$ of (2.1). Let us split the interval $[0, T]$ into r equal parts by the points

$\tau_k^* = k\Delta\tau$ $(k = 0, \ldots, r)$. We shall compute $w(\tau_k^*, l) = w_k$ where the function $w(\tau, l)$ is determined by Equation (2.3). Let us arrange the numbers w_k in decreasing order $w_{k_1} \geqslant w_{k_2} \geqslant \ldots \geqslant w_{k_r}$.

The number s is determined from the condition

$$s = E\left\{\frac{\mu(\Delta)}{\Delta\tau}\right\}$$

Then the system of intervals Δ is approximately

$$[\tau_{kj}^*, \tau_{kj}^* + \Delta\tau] \quad (j = 1, \ldots, s)$$

The value of the function (2.1) is determined by Equation

$$\rho(l) \approx \Delta\tau \sum_{j=1}^{s} w_{kj}$$

In a similar manner the quantities $\partial\rho/\partial l_i$ $(i = 1, \ldots, n)$ are determined. The accuracy of the computation is improved as the number r increases.

The method exposed for the computation of the intervals Δ can be easily set up on a digital computer.

3. Let us consider some particular problems which may be solved by the method exposed in Section 1 of the present paper.

Problem 3.1. Let $u^\circ(t, \theta)$ be the optimum control for the problem of Section 1. Find a value $\theta = \theta^*$ of the parameter appearing in the functional (1.2) such that the optimum control $u^\circ(t, \theta^*)$ satisfy the additional condition

$$\max_\tau |u^\circ(\tau, \theta^*)| = H$$

in which H is a given constant number.

From the method [1] used to determine $\max_\tau |u^\circ(\tau, \theta)|$ follows the continuous and monotonous dependence of this value on the parameter θ

It follows that the problem (3.1) can be solved if there exist two values θ_1 and θ_2 of the parameter θ for which the condition

$$\max_\tau |u^\circ(\tau, \theta_1)| < H < \max_\tau |u^\circ(\tau, \theta_2)|$$

is fulfilled.

In that case the approximate determination of θ^* can be, for instance, reduced, first to the division of the segment $[\theta_1, \theta_2]$ and then to the solution of the problem of Section 1 for the values of θ which are found.

Problem 3.2. Often the control possibilities of the system are limited. This means that the motor which develops a certain force, can work only during a certian length of time. Therefore, it is interesting to determine the domain in which the initial conditions of the system (1.1) should lie so that, from any of these points, an optimum control $u^\circ(t)$ could be found such that it brings, in the time T, the system to the origin of the coordinates, and gives a minimum of (1.2) with the condition that the motor develops a force $|u| \leqslant H$ during the time $\mu(\Delta) = 1/\theta < T$. This problem reduces to the problem: find the domain of the possible values of the vector x_0 for which

$$\min_l \max_\Delta \int_\Delta \left| \sum_{i=1}^{n} l_i h_i(\tau) \right| d\tau \geqslant \frac{1}{H} \tag{3.1}$$

with the condition

$$c_i = -x_{i0}, \qquad \sum_{i=1}^{n} l_i c_i = 1, \qquad \mu(\Delta) = \frac{1}{\theta}, \qquad h(\tau) = F(-\tau)B$$

Let us get an estimate of the sought for domain. Let us denote the left-hand side of (3.1) by $G(x)$. We shall find the value of $G(x)$ for the points

$$x^{(1)}\ (a^{-1}, 0, \ldots, 0),\ x^{(2)}\ (0, a^{-1}, \ldots, 0), \ldots, x^{(n)}(0, \ldots, a^{-1})\ (a > 0)$$

We shall denote

$$G(x^i) = G_i \quad (i = 1, \ldots, n), \qquad d = \min\ \{G_i\}$$

The hyperplanes

$$\sum_{i=1}^{n} l_i c_i^{\,j} = 1 \qquad (j = 1, \ldots, n) \tag{3.2}$$

corresponding to the points x^j represent a n-dimensional parallelepiped in the domain l

$$\prod_{i=1}^{n} (a^2 - l_i^2) = 0 \tag{3.3}$$

The maximum distance of the points of this parallelepiped to the origin of the coordinated is obviously

$$\rho = a \sqrt{n} \tag{3.4}$$

The distance from the hyperplanes (3.2) to the origin of the coordinates of the space l is determined by the quantity

$$R = \frac{1}{\sqrt{c_1^2 + \cdots + c_n^2}} = \frac{1}{\|x\|_2} \tag{3.5}$$

Here the symbol $\|x\|_2$ represents the modulus of the vector x. Let the point x satisfy the condition

$$\|x\|_2 \leqslant \frac{1}{a\sqrt{n}} \tag{3.6}$$

We shall prove that for the x satisfying the inequality (3.6),

$$G(x) \geqslant d$$

Let x be any arbitrary point, satisfying the inequality (3.6). On the basis of (3.6), the corresponding vector l will intersect some face of the parallelepiped (3.3). Therefore this vector can be represented in the form $l = \eta l'$ where $\eta \geqslant 1$, and the vector l' ends on the edge of the parallelepiped (3.3). Therefore

$$G(x) = \eta \int_\Delta \Big| \sum_{i=1}^{n} l_i' h_i(\tau) \Big|\, d\tau \geqslant \int_\Delta \Big| \sum_{i=1}^{n} l_i' h_i(\tau) \Big|\, d\tau > d$$

The number d can be modified by the choice of the number a. Let us consider the number λa. The new value of the quantities G_i will then be λG_i and the new value of d is λd.

Assuming $a = 1$, $\lambda d = 1/H$ and taking (3.6) into consideration, we get the sought estimate

$$\|x\|_2 \leqslant \frac{Hd}{\sqrt{n}} \tag{3.7}$$

4. Let us consider the following illustrative examples.

Example 4.1. Let the motion of the control system be described by the differential equations

$$x_1^{\cdot} = x_2, \qquad x_2^{\cdot} = -\alpha x_1 + \beta x_3, \qquad x_3^{\cdot} = u \tag{4.1}$$

Let us determine the control $u(t)$ which brings the system (4.1) in the time $0 \leqslant t \leqslant T$ to its equilibrium position $(x_1 = x_2 = x_3 = 0)$ in a manner such that the functional (1.2) has a minimum value. We shall solve the problem for the following numerical values:

$$\alpha = 14 \cdot 10^{-7}, \qquad \beta = 3 \cdot 10^{-3}, \qquad T = 5360, \qquad \theta = {}^1/_{134}$$

The initial position of the system (4.1) is given by

$$x_{10} = 37 \cdot 10^{-3}, \qquad x_{20} = 0, \qquad x_{30} = 0 \tag{4.2}$$

The fundamental solution matrix of the homogeneous system (4.1) has the form

$$F(t) = \begin{pmatrix} \cos at & a^{-1}\sin at & b(1-\cos at) \\ -a\sin at & \cos at & ab\sin at \\ 0 & 0 & 1 \end{pmatrix} \qquad (4.3)$$

Here $a = \sqrt{\alpha} = 1.17\cdot 10^{-3}$; $b = \beta / \alpha = 2\cdot 19$. The function $h_i(\tau)$ $(i = 1, 2, 3)$ have the form

$$h_1(\tau) = b(1-\cos a\tau), \quad h_2(\tau) = -ab\sin a\tau, \quad h_3(\tau) = 1 \qquad (4.4)$$

The numbers c_i are given by

$$c_1 = -37\cdot 10^{-3}, \qquad c_2 = 0, \qquad c_3 = 0 \qquad (4.5)$$

In agreement with Section 1 of the present paper, the sought control $u^\circ(t)$ is given by the solution of the problem

$$\min_l \max_\Delta \int_\Delta |l_1 b(1-\cos a\tau) - l_2 ab\sin a\tau + l_3|\, d\tau = \gamma \qquad (4.6)$$

with the condition

$$l_1c_1 + l_2c_2 + l_3c_3 = 1. \qquad \text{mes}\,\Delta = 134$$

The solution of the problem (4.6) calculated on a digital computer «Урал - 2» (Ural - 2) by the method of steepest descent was in agreement with the result of the Section 2 of the present paper. The following results were obtained:

$$\gamma = 7930, \qquad l_1^\circ = -27, \qquad l_2^\circ = 1.03, \qquad l_3^\circ = 59$$

The system of intervals Δ° was determined by

$$[0,34], \qquad [2646, 2713], \qquad [5326, 5360]$$

Thus, the optimum control $u^\circ(\tau)$ found on the basis of (1.4) is defined by

$$u^\circ(\tau) = 0.126\cdot 10^{-3}\,\text{sign}\cos a\tau \quad \text{for } \tau \text{ on } \Delta^\circ, \qquad u^\circ(\tau) = 0 \quad \text{for } \tau \text{ outside } \Delta^\circ \qquad (4.7)$$

a graphical representation of the optimum control (4.7) is shown on Fig.4.

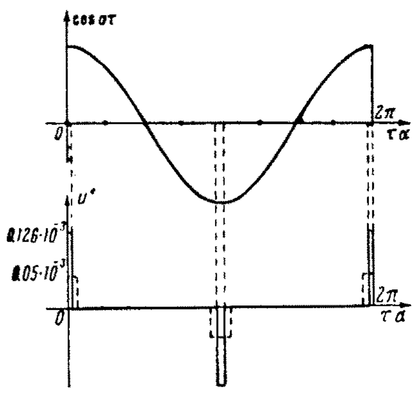

Fig. 4

Let us note also that we could consider by this method, the problem of the plane motion correction of a material point on a near-circular orbit in an equatorial plane [3], if the problem is considered in its linear approximation. For an unbounded increase in θ, the solution of the problem is similar to the impulsive control analogous to that considered in [3]. It must be mentioned however, that unlike in [3], the problem has been considered only in its linear approximation.

E x a m p l e 4.2 . Let it now be required to find, in the functional (1.2) a value of θ such that the optimum control of Example (4.1) satisfies the complementary condition

$$\max_\tau |u(\tau)| = 5\cdot 10^{-5} \qquad (4.8)$$

In agreement with Section 3 of the present paper, the problem (4.6) was solved for the following values of the parameter θ : θ = 403, 348, 335. Thus the following values were found for the numbers $1/\gamma$:

$$1/\gamma = 4.23\cdot 10^{-5},\ 4.87\cdot 10^{-5},\ 5.05\cdot 10^{-5}$$

The value of the parameter θ , for which the condition (4.8) was satisfied, was found to be

$$\theta^* = 338.$$

The optimum control in that case is determined by the expressions

$$u^\circ(\tau) = 5\cdot 10^{-5}\,\mathrm{sign}\cos a\tau \quad \text{for } \tau \text{ on } \Delta^\circ, \qquad u^\circ(\tau) = 0 \quad \text{for } \tau \text{ outside } \Delta^\circ \qquad (4.9)$$

thus the system of intervals Δ° is determined as

$$[0, 84], \quad [2595, 2764], \quad [5275, 5360]$$

The graph of the control function (4.9) which was found is shown by a dotted line on Fig.4.

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